Analytic lifts of operator monotone and concave functions

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October 21, 2020

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Introduction

In this talk, E will denote a Hilbert space

- $\mathbb{S}(E)$ denote the space of self-adjoint operators
- $\mathbb{P} \subseteq \mathbb{S}$ denotes the cone of invertible positive definite and $\hat{\mathbb{P}}$ the cone of positive semi-definite operators

$\mathbb{S}$ and $\mathbb{P}$ are partially ordered cones with the positive definite order:

$$A \leq B \text{ iff } B - A \text{ is positive semidefinite}$$

Definition

A real function $f : (0, \infty) \mapsto \mathbb{R}$ is operator monotone, if $A \leq B$ implies $f(A) \leq f(B)$ for $A, B \in \mathbb{P}(E)$ and all E.

Some examples:

- x^t for $t \in [0, 1]$;
- $\log x$;
- $\frac{x-1}{\log x}$.

Loewner's theorem

Theorem (Loewner 1934)

A real function $f : (0, \infty) \mapsto \mathbb{R}$ *is operator monotone if and only if*

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + x} d\mu(\lambda),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ *and* μ *is a unique non-negative measure on* $[0, \infty)$ *such that* $\int_0^\infty \frac{1}{\lambda^2+1} d\mu(\lambda) < \infty$; *if and only if it has an analytic continuation to the open upper complex half-plane* $\mathbb{H}^+$, *mapping* $\mathbb{H}^+$ *to* $\mathbb{H}^+$.

Many different proofs of Loewner's theorem exist:

- Bendat-Sherman '55, Hansen '13, Hansen-Pedersen '82, Korányi-Nagy '58, Sparr '90, Wigner-von Neumann '54, ...

Free and matrix convex sets

$\mathcal{A}$ a Banach space, $I_{\mathcal{A}} : \mathcal{A} \mapsto \mathcal{A}$ denote the identity map.

Definition (Free set and matrix convex set)

A collection $(D(E))$ of sets of operators $D(E) \subseteq \mathcal{A} \otimes \mathcal{B}(E)$ for each Hilbert space E is a *free set* whenever for all Hilbert spaces E, K we have the following:

(1) $(I_{\mathcal{A}} \otimes U^*)D(E)(I_{\mathcal{A}} \otimes U) \subseteq D(K)$ for all unitary $U : K \mapsto E$.

(2) $D(E) \oplus D(K) \subseteq D(E \oplus K)$.

If additionally (1) holds for any linear isometry $U : K \mapsto E$, then $(D(E))$ is a *matrix convex set*.
Sometimes the collection $(D(E))$ will be restricted to the case $\dim(E) < \infty$. In that case, for all other involved Hilbert spaces K we assume $\dim(K) < \infty$ as well.

Free functions

Non-commutative polynomials of k operator variables suggest:

Definition (Free function)

Let $\mathcal{L}$ be a fixed Hilbert space. A collection of functions $F : D(E) \mapsto \mathcal{B}(\mathcal{L} \otimes E)$ indexed by E for a free set $D(E) \subseteq \mathcal{A} \otimes \mathcal{B}(E)$ defined for all Hilbert spaces E, K is called a *free function* whenever for all $A \in D(E)$ and $B \in D(K)$, we have

(1) unitary invariance, that is

$$F((I_{\mathcal{A}} \otimes U^*)A(I_{\mathcal{A}} \otimes U)) = (I_{\mathcal{L}} \otimes U^*)F(A)(I_{\mathcal{L}} \otimes U)$$

holds for all unitaries $U : E \mapsto K$;

(2) direct sum invariance, that is

$$F(A \oplus B) = F(A) \oplus F(B).$$

Operator monotone, concave functions

Definition (Operator monotonicity)

An free function $F : D(E) \mapsto \mathcal{B}(E)$ is operator monotone if for all $X, Y \in D(E)$ s.t. $X \leq Y$, we have

$$F(X) \leq F(Y).$$

If this property is verified only (hence up to) $\dim(E) = n$, then F is n-monotone. Example: Karcher mean, ALM, BMP, etc.

Definition (Operator concavity & convexity)

A free function $F : D(E) \mapsto \mathcal{B}(E)$ is operator concave if for all $X, Y \in D(E)$ and $\lambda \in [0, 1]$, we have

$$(1-\lambda)F(X) + \lambda F(Y) \leq F((1-\lambda)X + \lambda Y)$$

Similarly we define n-concavity. Note: convexity of $D(E)$ is needed.

Recent multivariable results

- For an operator convex free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ that is rational - hence already free analytic and defined for general tuples of operators by virtue of non-commutative power series expansion - Helton, McCullogh and Vinnikov in 2006, JFA proved a representation formula, that is superficially similar to our formula that we will obtain here later in full generality.
- For a real valued continuously differentiable locally monotone function $f : \mathbb{R}^k \mapsto \mathbb{R}$ Agler, McCarthy and Young in 2012, Ann. Math. proved a representation formula valid for pairwise commuting tuples of operators. Using the formula they obtained the analytic continuation of the restricted f to $(H^+)^k$ mapping $(H^+)^k$ to H^+.

Recent multivariable results

- In JFA 2017, Pascoe and Tully-Doyle proved that a free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ that is free analytic, i.e. has a non-commutative power series expansion, thus already defined for general tuples of operators, is operator monotone if and only if it maps the upper operator poly-halfspace $\Pi(E)^k$ to $\overline{\Pi}(E)$ for all finite dimensional E, where $\Pi(E) := \{X \in \mathcal{B}(E) : \frac{X-X^*}{2i} > 0\}$.
- Recently in LAA 2018, Pascoe provided an alternative argument using the Agler-McCarthy-Young theorem leading to analytic continuation, which has been relaxed in 2019 by Pascoe and Tully-Doyle using the original Loewner theorem.

Note: matrix convexity of $(D(E))$ is crucial in the above results, except in Agler-McCarthy-Young.

Motivation of non-matrix convex domains through means

In 2012 P-Lim introduced the matrix power means as non-commutative operator monotone lifts of the power means

$$P_p(\omega; A) := \left(\sum_{i=1}^{k} w_i A_i^p\right)^{1/p}$$

of positive definite commuting k-tuples $A = (A_1, \ldots, A_k)$, a probability vector ω and $p \in [-1, 1]$. The idea is to consider the unique positive definite solution $X =: P_p(\omega; A)$ of

$$\sum_{i=1}^{k} w_i X \#_p A_i = X.$$

$P_p(\omega; \cdot)$ is operator monotone and as $p \to 0$, it converges to the multivariable geometric (Karcher) mean $\Lambda(\omega; \cdot)$.

A characterization of concavity

Proposition (P 2020)

Let $(D(E))$ *with* $D(E) \subseteq \mathcal{A} \otimes \mathcal{B}(E)$ *denote a self-adjoint matrix convex set and let* $F : D(E) \mapsto \mathcal{B}(E)$ *be a free function. Then* F *is operator concave if and only if for each isometry* $W : E \mapsto K$ *and* $X \in D(K)$ *we have*

$$F\left((I_{\mathcal{A}} \otimes W^*)X(I_{\mathcal{A}} \otimes W)\right) \geq W^* F(X) W.$$

Corollary (P 2020)

Under the above assumptions if also $0 \in D(\mathbb{C})$ *and* $F(0) \geq 0$*, then the equivalence remains true with contractions* $W : E \mapsto K$*.*

Given a disjoint union of sets $(C(E) \subseteq \mathcal{A} \otimes \mathcal{B}(E))$ for each Hilbert space E closed under direct sums, its matrix convex hull is given as

$$\mathrm{co}^{\mathrm{mat}} C(E) := \bigcup_K \{V^* X V : X \in C(K), V : E \mapsto K \text{ an isometry}\}.$$

If $0 \in C(\mathbb{C})$ then we also have

$$\mathrm{co}^{\mathrm{mat}} C(E) = \bigcup_K \{V^* X V : X \in C(K), V : E \mapsto K, \|V\| \leq 1\}.$$

Given a collection of sets $(D(E) \subseteq \mathcal{A} \otimes \mathcal{B}(E))$ closed under direct sums and a collection of functions $F : D(E) \mapsto \mathcal{B}(E)$ preserving direct sums, we consider its hypograph

$$\mathrm{hypo}(F) := (\mathrm{hypo}(F)(E)) := (\{(Y, X) \in \mathcal{B}(E) \times D(E) : Y \leq f(X)\}).$$

A characterization of partial concavity

Proposition (P 2020)

Let a collection of self-adjoint sets $(D(E) \subseteq \mathcal{A} \otimes \mathcal{B}(E))$ closed under direct sums and a collection of functions $F : D(E) \mapsto \mathcal{B}(E)$ preserving direct sums be given. Then for each isometry $W : E \mapsto K$ and $X \in D(K)$ such that $(I_{\mathcal{A}} \otimes W^*)X(I_{\mathcal{A}} \otimes W) \in D(E)$ we have that

$$F((I_{\mathcal{A}} \otimes W^*)X(I_{\mathcal{A}} \otimes W)) \geq W^* F(X) W,$$

if and only if for each $(Y, X) \in \mathrm{co}^{\mathrm{mat}}(\mathrm{hypo}(F))(E)$ with $X \in D(E)$ we have that $Y \leq F(X)$.
Moreover if $0 \in D(\mathbb{C})$ and $F(0) \geq 0$ then the statement holds with contractions $W : E \mapsto K$.

Supporting pencils of hypographs

Proposition (P 2020)

Let $(D(E)) \ni 0$ *and* F *be as before with* $F|_D > 0$*. Assume that* $\mathrm{co}^{\mathrm{mat}}(D)(E)$ *has nonempty interior for each* E*. Let* N *be a Hilbert space. Then for each interior point* $A \in D(N)$ *and each unit vector* $v \in N$ *there exists a completely bounded affine linear map* $L_{F,A,v} : (\mathcal{B}(E), \mathcal{A} \otimes \mathcal{B}(E)) \mapsto \mathcal{B}(N)^* \otimes \mathcal{B}(E)$ *given as*

$$L_{F,A,v}(Y, X) := T(F, A, v) \otimes I_E - vv^* \otimes Y + \Lambda_{F,A,v}(X),$$

where $0 \leq T(F, A, v) \in \mathcal{B}(N)^*$ *and* $\Lambda_{F,A,v} : \mathcal{A} \mapsto \mathcal{B}(N)^*$ *is a self-adjoint completely bounded linear map, such that*

(a) $T(F, A, v)(I_N) = v^*F(A)v - \Lambda_{F,A,v}(A)$ *and there exists* $\epsilon > 0$ *such that* $(1 + \epsilon)A \in \mathrm{co}^{\mathrm{mat}}(D)(N)$ *and* $-\Lambda_{F,A,v}(A) \leq \frac{v^*F(A)v - v^*F((1+\epsilon)A)v}{\epsilon}$*;*

(b) For all $(Y, X) \in \mathrm{hypo}(F)$ we have $L_{F,A,v}(Y, X) \geq 0$;

(c) $\gamma^* L_{F,A,v}(F(A), A)\gamma = 0$ where $\gamma = I_N$;

(d) For every X in the interior of $\mathrm{co}^{\mathrm{mat}}(D)(E)$ there exists an $\epsilon > 0$ such that $\langle V, L_{F,A,v}(0, X)V\rangle \geq \epsilon T(F, A, v)(V^* V)$.

Let $(D(E)) \ni 0$ be as above. Then for a Hilbert space E and a dense set $E_0 \in \{x \in E : \|x\| = 1\}$ define the auxiliary vector space

$$\mathcal{H}_{E,0} := \bigoplus_{(X,v)\in(D(E),E_0)} E$$

and its completion $\mathcal{H}_E$ with respect to the usual inherited direct sum inner product. We denote by $I_{(X,v)} \in \mathcal{B}(\mathcal{H}_E, E)$ the isometry that equals to $I_E - vv^*$ on the (X, v) slot and 0 elsewhere.

A representation formula

Corollary (P 2020)

Let $(D(E)) \ni 0$ *and* F *be as above with* $F|_D > 0$*. Fix a Hilbert space* E *and an* $\eta > 0$*. Assume that* $\mathrm{co}^{\mathrm{mat}}(D)(E)$ *has nonempty interior for* E*. Then there exists a vector* $e \in \mathcal{H}_E$ *with* $\|e\| = 1$*, a completely bounded affine map* $L_F : \mathcal{A} \otimes \mathcal{B}(E) \mapsto \mathcal{B}(\mathcal{H}_E)^* \otimes \mathcal{B}(E)$ *given as*

$$L_F(X) := T_F \otimes I_E + \Lambda_F(X),$$

where $0 \leq T_F \in \mathcal{B}(\mathcal{H}_E)^*$ *and* $\Lambda_F : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_E)^*$ *is self-adjoint completely absolutely continuous with respect to* T_F*, such that*

(a) *For all* $X \in \mathrm{co}^{\mathrm{mat}}(D)(E)$ *we have* $L_F(X) \geq 0$*;*

(b) *For all* $(1+\eta)X \in D(E)$ *in the interior of* $\mathrm{co}^{\mathrm{mat}}(D)(E)$*,* $W \in \overline{\mathcal{B}(\mathcal{H}_E, E)_{T_F}}$ *and* $v \in E$ *we have*

$$\left\langle W, L_F(X)(I_{(X,v)} + ve^*)\right\rangle_{T_F} = e^* W^* F(X) v.$$

Let $Z \geq 0$ and S a subspace. Let $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ with $Z_{11} : S \mapsto S$, $Z_{21} : S \mapsto S^{\perp}$. Then $\operatorname{ran}(Z_{21}) \subset \operatorname{ran}(Z_{22})^{1/2}$ and $\exists C : S \mapsto S^{\perp}$ such that $Z_{21} = (Z_{22})^{1/2} C$ and

$$Z = \begin{bmatrix} Z_{11} - C^*C & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C^* & 0 \\ (Z_{22})^{1/2} & 0 \end{bmatrix} \begin{bmatrix} C & (Z_{22})^{1/2} \\ 0 & 0 \end{bmatrix}.$$

Then $\mathcal{S}_S(Z) := Z_{11} - C^*C$ is maximal among $X : S \mapsto S, X \leq Z$.

Theorem (P 2020)

Let $(D(E)) \ni 0$ and F be as above with $F|_D > 0$. Fix E. Assume that $\mathrm{co}^{\mathrm{mat}}(D)(E)$ has nonempty interior for E. Then for each $X \in D(E)$ in the interior of $\mathrm{co}^{\mathrm{mat}}(D)(E)$ we have

$$F(X) = (e \otimes I_E)\mathcal{S}_{e^* \otimes E}(L_F(X))(e^* \otimes I_E).$$

Moreover the RHS is well defined for each interior point $X \in \mathrm{co}^{\mathrm{mat}}(D)(E)$.

Theorem (P 2020)

Let $\mathcal{H}$ a Hilbert space and $e \in \mathcal{H}$ with $\|e\| = 1$ be fixed. Let a completely bounded affine map $L : \mathcal{A} \otimes \mathcal{B}(E) \mapsto \mathcal{B}(\mathcal{H})^* \otimes \mathcal{B}(E)$ be given as

$$L(X) := T \otimes I_E + \Lambda(X),$$

where $0 \leq T \in \mathcal{B}(\mathcal{H})^*$ and $\Lambda : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H})^*$ is a self-adjoint completely bounded linear map that is completely absolutely continuous with respect to T. Then the function

$$F(X) := (e \otimes I_E)\mathcal{S}_{e^* \otimes E}(L(X))(e^* \otimes I_E)$$

is well defined and analytic for each $X \in \{Y \in \mathcal{A} \otimes \mathcal{B}(E) : L(\Re(Y)) > 0\}$ and satisfies the assumptions of the above with $D(E) := \{Y \in \mathcal{A} \otimes \mathcal{B}(E) : L(Y) \geq 0\}$.

Free lifting of globally operator monotone functions

Let $\mathbb{CP}(E)^k$ denote the set of pairwise commuting k-tuples of positive operators on E. A real function $f:(0,\infty)^k \mapsto (0,\infty)$ has a natural functional calculus on $\mathbb{CP}(E)^k$ for $\dim(E)<\infty$ defined as $f(X):=U^*f(\Lambda)U$ where $X=U^*\Lambda U$ denotes the joint spectral decomposition of $X\in\mathbb{CP}(E)^k$ and $f(\Lambda):=\bigoplus_{i=1}^k f(\{\Lambda_1\}_{ii},\ldots,\{\Lambda_k\}_{ii})$. We also have $\mathrm{co}^{\mathrm{mat}}\mathbb{CP}^k(E)=\mathbb{P}^k(E)$ for each $\dim(E)<\infty$.

Definition

A real function $f:\mathbb{P}(\mathbb{C})^k\mapsto\mathbb{P}(\mathbb{C})$ is said to be *(globally) operator monotone*, if for any $X\leq Y\in\mathbb{CP}(E)^k$, $\dim(E)<+\infty$ we have $f(X)\leq f(Y)$.

Problem

Is every real globally operator monotone function admit an analytic continuation to $\mathbb{P}(E)^k$ or a global formula?

Proposition (P 2020)

Let $f : \mathbb{P}(\mathbb{C})^k \mapsto \mathbb{P}(\mathbb{C})$ *be a (globally) operator monotone function. Then for any isometry* $W : E \mapsto K$ *between finite dimensional Hilbert spaces* E, K *and any* $X \in \mathbb{CP}(K)^k$ *such that* $W^* X W \in \mathbb{CP}(E)^k$ *we have*

$$W^* f(X) W \leq f(W^* X W).$$

In particular f *is concave and continuous as a real function.*

Theorem (P 2020)

Let $f : \mathbb{P}(\mathbb{C})^k \mapsto \mathbb{P}(\mathbb{C})$ *be a real function. Then* f *is (globally) operator monotone if and only if for each* $(Y, X) \in \mathrm{co}^{\mathrm{mat}}(\mathrm{hypo}(f)(E))$ *with* $\dim(E) < +\infty$ *and* $X \in \mathbb{CP}(E)^k$*, we have that* $Y \leq f(X)$*.*

Thus, we can follow the earlier arguments, obtaining analytic shorted operator based formulas!

Operator means of probability measures

Let $\mathcal{P}(\mathbb{P}(E))$ denote the set of fully supported Borel probability measures on the complete metric space $(\mathbb{P}(E), d_\infty)$ where E is a Hilbert space and $d_\infty(A,B) = \|\log(A^{-1/2}BA^{-1/2})\|$ denotes the Thompson metric. Let $\mathcal{P}^\infty(\mathbb{P}(E)) \subset \mathcal{P}(\mathbb{P}(E))$ denote the subset of probability measures with bounded support. For a $\mu \in \mathcal{P}(\mathbb{P}(E))$ the support $\mathrm{supp}(\mu)$ is a separable closed subset of $\mathbb{P}(E)$ and it has full measure $\mu(\mathrm{supp}(\mu)) = 1$.
A set $U \subseteq \mathbb{P}(E)$ is *upper* if $X \leq Y \in \mathbb{P}(E)$ and $X \in U$ imply that $Y \in U$.

Definition (Stochastic order)

For $\mu, \nu \in \mathcal{P}(\mathbb{P}(E))$ the stochastic partial order $\mu \leq \nu$ is defined by requiring $\mu(U) \leq \nu(U)$ for all closed upper sets $U \subseteq \mathbb{P}(E)$.

Idea: utilize the Skorokhod representation

Stochastic order of probability measures

Theorem (Strassen 1965)

Let $\mu, \nu \in \mathcal{P}^\infty(\mathbb{P}(E))$. Then the following are equivalent:

(i) $\mu \le \nu$;

(ii) *there exists* $\xi_\mu : [0,1] \mapsto \operatorname{supp}(\mu)$ *and* $\xi_\nu : [0,1] \mapsto \operatorname{supp}(\nu)$ *such that* $\mu = (\xi_\mu)_*\lambda$ *and* $\nu = (\xi_\nu)_*\lambda$ *with* $\xi_\mu(t) \le \xi_\nu(t)$ *almost surely for all* $t \in [0,1]$.

Proposition (P 2020)

The collection of sets $(\mathcal{P}^\infty(\mathbb{P}(E)))$ *indexed by* E *is a self-adjoint matrix convex set. In particular* $\mathcal{P}^\infty(\mathbb{P}(E))$ *embeds into* $L^\infty([0,1],\lambda)_+ \otimes \mathbb{P}(E)$*, the strictly positive cone of the projective tensor product* $L^\infty([0,1],\lambda) \otimes \mathcal{B}(E)$.

Nuclearity ensures that all C^*-cross norms on $(L^\infty([0,1],\lambda)^+ \otimes \mathbb{P}(E))$ are equivalent.

Definition (Direct sums of probability measures)

For $\mu \in \mathcal{P}^\infty(\mathbb{P}(E))$, $\nu \in \mathcal{P}^\infty(\mathbb{P}(K))$, let $\Gamma(\mu, \nu) \subseteq \mathcal{P}^\infty(\mathbb{P}(E \oplus K))$ denote the set of couplings of μ, ν, that is $\gamma \in \Gamma(\mu, \nu)$ if $\gamma(A \times \mathbb{P}(K)) = \mu(A)$ and $\gamma(\mathbb{P}(E) \times B) = \nu(B)$. Then $\mu \oplus \nu$ is defined to be the set $\Gamma(\mu, \nu)$. Thus in general, the direct sum of probability measures is no longer uniquely determined.

Notice that $\Gamma(\mu, \nu)$ is nonempty, since $\mu \times \nu \in \Gamma(\mu, \nu)$.

Definition (Operator mean of discrete probability measures)

For each $0 < n \in \mathbb{N}$ and E let $F_n : \mathbb{P}(E)^n \mapsto \mathbb{P}(E)$ be an operator monotone free function. Then $F = \{F_n\}$ is an operator mean if

1) For a permutation $\sigma \in S_n$,
$F_n(X_1, \ldots, X_n) = F_n(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$;
2) For $0 < k \in \mathbb{N}$,
$F_{nk}(\underbrace{X_1, \ldots, X_1}_{k \text{ times}}, \ldots, \underbrace{X_n, \ldots, X_n}_{k \text{ times}}) = F_n(X_1, \ldots, X_n)$.

Proposition (P 2020)

An operator mean $F_n : \mathbb{P}(E)^n \mapsto \mathbb{P}(E)$ *preserves direct sums of discrete probability measures with rational weights.*

In order to study operator means of general probability measures $F : \mathcal{P}^\infty(\mathbb{P}(E)) \mapsto \mathbb{P}(E)$, we consider first free functions of random variables, that is $F : (L^1([0,1],\lambda)^+ \otimes \mathbb{P}(E)) \mapsto \mathbb{P}(E)$. Let $S([0,1],\lambda)$ denote the set of simple functions on $[0,1]$. Then $S([0,1],\lambda)$ is norm-dense in $L^p([0,1],\lambda)$ for $1 \leq p \leq +\infty$ and the same is true for $S([0,1],\lambda)^+ \otimes \mathbb{P}(E)$ in $L^p([0,1],\lambda)^+ \otimes \mathbb{P}(E)$.

Theorem (P 2020)

Assume that $F : S([0,1],\lambda)^+ \otimes \mathbb{P}(E) \mapsto \mathbb{P}(E)$ *is free operator concave function. Then for each* $1 \leq p \leq +\infty$ *there exists a unique* $\hat{F}_p : L^p([0,1],\lambda)^+ \otimes \mathbb{P}(E) \mapsto \mathbb{P}(E)$ *extending* F.

Extension and representation theorems for operator means

Theorem (P 2020)

Assume that the sequence of functions $F_n : \mathbb{P}(E)^n \mapsto \mathbb{P}(E)$ *for* $0 < n \in \mathbb{N}$ *is an operator mean of discrete probability measures. Then it uniquely extends into a stochastic order preserving function* $\hat{F} : \mathcal{P}^\infty(\mathbb{P}(E)) \mapsto \mathbb{P}(E)$.

Corollary (P 2020)

Let $F : \mathcal{P}^\infty(\mathbb{P}(E)) \mapsto \mathbb{P}(E)$ *be a stochastic order preserving free function. Then there exists an operator monotone free function* $\hat{F} : L^\infty([0,1],\lambda)^+ \otimes \mathbb{P}(E) \mapsto \mathbb{P}(E)$ *that represents* F *and* $\hat{F}(X + I)$ *is given by the shorted operator formula where* $0 \leq \lambda \in (L^\infty([0,1],\lambda) \otimes \mathcal{B}(\mathcal{H}_E))^*$ *and* $I(t) := I_E$ *for all* $t \in [0,1]$.

Blecher's real operator monotonicity

For a free function $F : D(E) \mapsto \mathcal{B}(\mathcal{L} \otimes E)$ we define real operator monotonicity of F as: $A \leq_{\mathrm{Re}} B$ implies $F(A) \leq_{\mathrm{Re}} F(B)$, where $0 \leq_{\mathrm{Re}} X$ means $0 \leq \Re X = \frac{X+X^*}{2}$. Note: $\leq_{\mathrm{Re}}$ is just a preorder.

Theorem (Gaál & P 2020)

Let the domain $(D(E))$ be open. Then F is real operator monotone if and only if

$$F(\Re X, \Im X) = G(\Re X) + \mathrm{i} H(\Re X, \Im X)$$

where $H : \Re D(E) \times \Im D(E) \mapsto \mathbb{S}(E)$ is a free function and $G : \Re D(E) \mapsto \mathbb{S}(E)$ is an operator monotone free function.

Assume further that for all invertible $S \in \mathcal{B}(E, K)$

(1') $F((S^{-1} \otimes I_{\mathcal{A}})A(S \otimes I_{\mathcal{A}})) = (I_{\mathcal{B}(\mathcal{L})} \otimes S^{-1})F(A)(I_{\mathcal{B}(\mathcal{L})} \otimes S)$.

Note that: (1'), (2) + local boundedness is equivalent to that F is free holomorphic! Then:

Theorem (Real monotone Loewner's theorem, Gaál & P 2020)

Let $F : D(E) \mapsto \mathcal{B}(\mathcal{L} \otimes E)$ be a free holomorphic function. Then F is real operator monotone if and only if

$$F(X) = C \otimes I + \phi(X)$$

where $C \in \mathcal{B}(\mathcal{L})$ and $\phi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{L})$ is completely positive linear.

Corollary (Gaál & P 2020)

Given a free set $(D(E))$, let $F : D(E) \to \mathcal{B}(E)$ be a free holomorphic function where each $D(E) \subseteq \mathcal{B}(E)^k$ is open. Then F is real operator monotone if and only if

$$F(X) = a_0 \otimes I + \sum_{j=1}^{k} a_j \otimes X_j$$

where $a_j \in \mathbb{C}$, with $a_j \geq 0$ for $j \in \{1, \ldots, k\}$.

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Thank you for your kind attention!