On the recent advances in the multivariable theory of operator monotone functions and means

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Introduction

In this talk, $E$ will denote a Hilbert space; $n, k$ are integers

- $\mathbb{S}(E)$ denote the space of self-adjoint operators over $E$
- $\mathbb{S}_n$ is its finite n-by-n dimensional part when $\text{dim}(E) = n$
- $\mathbb{P}(E) \subseteq \mathbb{S}(E)$ denotes the cone of invertible positive definite and $\hat{\mathbb{P}}(E)$ the cone of positive semi-definite operators
- $\mathbb{P}_n$ and $\hat{\mathbb{P}}_n$ denote the finite dimensional parts

$\mathbb{S}$ and $\mathbb{P}$ are partially ordered cones with the positive definite, also called Loewner order:

\[ A \leq B \text{ iff } B - A \text{ is positive semidefinite, i.e.} \]

\[ v^*(B - A)v \geq 0 \text{ for all } v \in E \]
Introduction

Loewner’s theorem

Definition
A real function $f : (a, b) \mapsto \mathbb{R}$ is called operator monotone if for all self-adjoint matrices $A \leq B$ with spectra included in $(a, b)$ we have $f(A) \leq f(B)$.

Theorem (Loewner 1934)
For a real function $f : (0, \infty) \mapsto \mathbb{R}$ the following are equivalent:

- $f$ is operator monotone;
- 
  
  
  $f(x) = \alpha + \beta x + \int_0^\infty \frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + x} d\mu(\lambda),$

  where $\alpha \in \mathbb{R}, \beta \geq 0$ and $\mu$ is a unique positive measure on $[0, \infty)$ such that $\int_0^\infty \frac{1}{\lambda^2 + 1} d\mu(\lambda) < \infty$;
- $f$ has an analytic continuation to the open upper complex half-plane $\mathbb{H}^+$, mapping $\mathbb{H}^+$ to $\mathbb{H}^+$.
**Operator connections & means**

**Definition (Kubo-Ando connection)**

A two-variable function $M: \hat{P} \times \hat{P} \rightarrow \hat{P}$ is called an operator connection if

1. if $A \leq A'$ and $B \leq B'$, then $M(A, B) \leq M(A', B')$,
2. $CM(A, B)C \leq M(CAC, CBC)$ for all Hermitian $C$,
3. if $A_n \downarrow A$ and $B_n \downarrow B$ then $M(A_n, B_n) \downarrow M(A, B)$,

where $\downarrow$ denotes the convergence in the strong operator topology of a monotone decreasing net.

- Additionally, if $M(A, A) = A$, then $M$ is called an operator mean.
Theorem (Kubo-Ando 1980, Math. Annalen)

An operator connection $M : \hat{P}^2 \mapsto \hat{P}$, is uniquely represented as

$$M(A, B) = A^{1/2} f \left( A^{-1/2} BA^{-1/2} \right) A^{1/2}$$

where $f : [0, \infty) \mapsto [0, \infty)$ is an operator monotone function.

Some operator connections on $\hat{P}^2$:

- Arithmetic mean: $\frac{A + B}{2}$
- Parallel sum: $A : B = (A^{-1} + B^{-1})^{-1}$
- Geometric mean: $A \# B = A^{1/2} (A^{-1/2} BA^{-1/2})^{1/2} A^{1/2}$

Kubo-Ando theory relies on Loewner’s theorem through the representing function $f$. It is not clear how to generalize the theory to multiple variables.
The Riemannian geometry of positive definite matrices

Assume that $E$ is finite dimensional. Then $\mathbb{P}(E)$ has a Riemannian structure with tangent space $\mathcal{S}(E)$:

$$
\langle X, Y \rangle_A = Tr \left\{ A^{-1}XA^{-1}Y \right\},
$$

$$
d^2(A, B) = \langle \log_A(B), \log_A(B) \rangle_A = Tr \left\{ \log^2(A^{-1/2}BA^{-1/2}) \right\},
$$

$$
\exp_A(X) = A^{1/2} \exp(A^{-1/2}XA^{-1/2})A^{1/2},
$$

$$
\log_A(B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2}
$$

for $X \in \mathcal{S}(E), A, B \in \mathbb{P}(E)$.

**Theorem**

The space $(\mathbb{P}(E), d)$ is an NPC or CAT(0)-space, i.e. for each pair $A, B \in \mathbb{P}(E)$ there exists a unique $Z \in \mathbb{P}(E)$, s.t. for all $X \in \mathbb{P}(E)$

$$
d^2(X, Z) \leq \frac{1}{2}d^2(X, A) + \frac{1}{2}d^2(X, B) - \frac{1}{4}d^2(A, B).
$$
Moreover in the above $Z = A\#B$, i.e. the geometric mean is the unique midpoint. Thus distance minimizing curves (geodesics) $\gamma : [0, 1] \mapsto \mathbb{P}(E)$ are unique between any pair $A, B \in \mathbb{P}(E)$ and the NPC-inequality extends to the whole curve

$$d^2(X, \gamma(t)) \leq (1 - t)d^2(X, A) + td^2(X, B) - t(1 - t)d^2(A, B)$$

where $\gamma(t) = A\#_t B := A^{1/2} (A^{-1/2}BA^{-1/2})^t A^{1/2}$ for $t \in [0, 1]$. The map $A\#_t B$ is called the weighted geometric mean. Thus points of the curve $A\#_t B$ admit the characterization

$$A\#_t B = \arg \min_{X \in \mathbb{P}} (1 - t)d^2(X, A) + td^2(X, B).$$

By Kubo-Ando $A\#_t B$ is monotone in the $A, B \in \hat{\mathbb{P}}(E)$ variables.
The multivariable geometric or Karcher mean

\[ \omega = (w_1, \ldots, w_n) \in \Delta_n \] probability vector, \( w_i > 0, \sum_{i=1}^{n} w_i = 1, \]
\[ \mathbb{A} = (A_1, \ldots, A_n), A_i \in \mathbb{P} \]

\[ \Lambda(\omega; \mathbb{A}) := \arg \min_{X \in \mathbb{P}} \sum_{i=1}^{n} w_i d^2(X, A_i) \]


- \( X \mapsto d^2(X, A) \) is a strictly geodesically convex function by the NPC property
- \( \Lambda(\omega; \mathbb{A}) \) is the unique solution of the (Riemannian) gradient equation called the Karcher equation:

\[ \nabla C(X) = -2 \sum_{i=1}^{n} w_i \log_{X} (A_i) = 0 \]

where \( C(X) = \sum_{i=1}^{n} w_i d^2(X, A_i) \).
Sturm’s law of large numbers

Theorem (Sturm 2002, Annals of Prob.)

Let \((X, d)\) be a CAT(0) space and let \(\mathcal{P}^2(X)\) denote the set of all probability measures \(\mu\) s.t. \(\int_X d^2(x, a) d\mu(a) < \infty\). Let \(a \#_t b\) denote the unique geodesic between \(a, b \in X\). Then for a probability measure \(\mu \in \mathcal{P}^2(X)\)

\[
\Lambda(\mu) := \arg \min_{x \in X} \int_X d^2(x, a) d\mu(a)
\]

exists and is unique. Moreover consider an i.i.d. sequence of random variables \(\{Y_i\}_{i \in \mathbb{N}}\) with law \(\mu\) and define the stochastic inductive sequence of geometric means

\[
S_1 := Y_1, \\
S_{k+1} := S_k \# \frac{1}{k+1} Y_{k+1}.
\]
The multivariable geometric or Karcher mean

*Then* $S_k$ *converges to* $\Lambda(\mu)$ *almost surely.*

Sturm’s result implies:


*For a fixed* $\omega = (w_1, \ldots, w_n) \in \Delta_n$ *the map* $\Lambda(\omega, \cdot)$ *is monotone,*

*i.e. for* $A_i, B_i \in \mathbb{P}(E), 1 \leq i \leq n$ *with* $A_i \leq B_i$ *we have*

$$\Lambda(\omega, A) \leq \Lambda(\omega, B).$$

There are other possible inductive extensions of the geometric mean $\#_t$ to multiple variables, namely iterative procedures defining the mean for $(n + 1)$-variables as a limit of sequences relying on up to $n$-variable versions of the same mean.
Other multivariable means

Some of these precede the Karcher mean like the ALM-geometric mean (Ando-Li-Mathias 2004, Linear Alg. Appl.) and the BMP-geometric mean (Bini-Meini-Poloni 2010, Math. Comp.). Other, more efficient iteration based variants can be found in

- (Hansen 2014, Linear Alg. Appl.);
- (Pálfia 2011, SIAM J. Matrix Anal Appl.);

and more. Some constructions extend arbitrary Kubo-Ando means to multiple variables, like the last two entries above.
Nodice theorem for the Karcher mean

There is a deterministic version of Sturm’s law of large numbers, first established for $\mathbb{P}(E)$ by (Holbrook 2012, J. Ramanujan M. S.):


Let $(X, d)$ be a CAT(0) space and let $\mu := \sum_{i=0}^{n-1} \frac{1}{n} \delta_{a_i}$ with $a_i \in X$. Consider the deterministic sequence $\{S_k\}_{k \in \mathbb{N}}$ defined as the inductive sequence of geometric means

$$S_1 := a_1,$$
$$S_{k+1} := S_k \# \frac{1}{k+1} a_{\overline{k}}$$

where $\overline{k} := k \mod (n)$. Then $S_k \to \Lambda(\mu)$ with rate $d(S_k, \Lambda(\mu)) = O(1/k)$.

The above along with Sturm’s law of large numbers even generalizes to CAT($\kappa$) spaces (Ohta-Pálfia 2015, Calc. Var. PDE).
The infinite dimensional case of positive operators

In the case of an infinite dimensional Hilbert space $E$, we no longer have a Riemannian structure so that the corresponding manifold is an NPC space. Thus the Karcher mean $\Lambda$ does not known to admit a definition as a unique solution of an optimization problem. However we still have the corresponding gradient equation called the Karcher equation:

$$\sum_{i=1}^{n} w_i \log_X(A_i) = 0$$

for a probability vector $\omega \in \Delta_n$ and operators $A_i \in \mathbb{P}(E)$ for $1 \leq i \leq n$. 


The Thompson or part metric

For $A, B \in \mathbb{P}(E)$ let $d_\infty(A, B) := \| \log(A^{-1/2}BA^{-1/2})\|$. 


The space $(\mathbb{P}(E), d_\infty)$ is a complete metric space and $d_\infty(A, B) = \log \max\{M(A \setminus B), M(B \setminus A)\}$ where $M(A \setminus B) := \inf\{\beta > 0 : B \leq \beta A\}$.

**Lemma (Lim-Pálfia 2012, J. Funct. Anal.)**

The map $f : \mathbb{P}(E) \mapsto \mathbb{P}(E)$ defined as

$$f(X) := \sum_{i=1}^{n} w_i X \#_t A_i$$

is a strict contraction in $(\mathbb{P}(E), d_\infty)$ for $t \in (0, 1]$ and $\omega \in \Delta_n$ and operators $A_i \in \mathbb{P}(E)$ for $1 \leq i \leq n$. That is $d_\infty(f(X), f(Y)) \leq (1 - t)d_\infty(X, Y)$. 

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On the recent advances in the multivariable theory of operator monotone functions and means

The infinite dimensional case of positive operators
The power means

By Banach fixed point theorem and some calculation:


For \( t \in [-1, 1] \), \( t \neq 0 \) and \( \omega \in \Delta_n \) and operators \( A_i \in \mathbb{P}(E) \) for \( 1 \leq i \leq n \), the nonlinear operator equation

\[
\sum_{i=1}^{n} w_i X \#_t A_i = X
\]

has a unique solution in \( \mathbb{P}(E) \) called the \( t \)-power mean denoted by \( P_t(\omega; A) \) which is monotone. Moreover it is monotone in \( t \), thus \( \{ P_t(\omega; A) \}_{t \in (0,1]} \) is a decreasing net, it has a greatest lower bound \( \Lambda(\omega, A) \), and the strong operator limit \( \lim_{t \to 0^+} P_t(\omega; A) = \Lambda(\omega, A) \).
The power means

The implicit function theorem with some trick implies:


The strong operator \( \lim_{t \to 0^+} P_t(\omega; A) = \Lambda(\omega, A) \) is the unique solution of the Karcher equation

\[
\sum_{i=1}^{n} w_i \log_X(A_i) = 0
\]

for \( X \in \mathbb{P}(E) \) and it is monotone in the second variable, i.e. if \( A_i \leq B_i \) for \( 1 \leq i \leq n \) and \( A_i, B_i \in \mathbb{P}(E) \), then \( \Lambda(\omega, A) \leq \Lambda(\omega, B) \).

If all \( A_i \) mutually commute for \( 1 \leq i \leq n \), then

\[
P_t(\omega; A) = \left( \sum_{i=1}^{n} w_i A_i^t \right)^{1/t}.
\]
The general theory of Lambda operator means

Let $\mathcal{P}^0(\mathbb{P}(E))$ denote the set of Borel probability measures with bounded support with respect to $d_\infty$ and let $\mathcal{L} := \{ f : (0, \infty) \mapsto \mathbb{R} \text{ is operator monotone, } f(1) = 0, f'(1) = 1 \}$. Consider the following real functions for $s, t \in [0, 1]$:

$$f_{s,t}(x) = \frac{[(1-t)(1-s)+t]x + s(1-t)}{(1-t)(1-s)x + t + s(1-t)}$$

for $t, s \in [0, 1]$.

Lemma (Pálfia 2016, Adv. Math.)

$f_{s,t}$ is operator monotone on $(0, \infty)$, is non-negative for all $s, t \in [0, 1]$ and

$$f_{s,t}(x) = l_s^{-1}( tl_s(x) ),$$

where $l_s(x) := \frac{x-1}{(1-s)x+s} \in \mathcal{L}$ and $l_s^{-1}$ is its inverse function.

Notation: let $M_{s,t}$ denote the Kubo-Ando mean represented by $f_{s,t}$. 
The general theory of Lambda operator means

Notation: $\mathcal{P}^0([0,1] \times \mathbb{P})$ is the simplex of Borel probability measures on $[0,1] \times \mathbb{P}$ with bounded support.

Proposition (Pálfia 2016, Adv. Math.)

Let $\mu \in \mathcal{P}^0([0,1] \times \mathbb{P}(E))$, $t \in (0,1]$. Then the equation

$$X = \int_{[0,1] \times \mathbb{P}} M_{s,t}(X, A) d\mu(s, A)$$

has a unique solution in $\mathbb{P}(E)$.

Definition (Induced Operator Mean)

Let $\mu \in \mathcal{P}^0([0,1] \times \mathbb{P}(E))$ and $t \in (0,1]$. Then we denote by $L_t(\mu)$ the unique solution $X \in \mathbb{P}(E)$ of the equation

$$X = \int_{[0,1] \times \mathbb{P}} M_{s,t}(X, A) d\mu(s, A).$$
The general theory of Lambda operator means

Definition (Generalized Karcher equation)
Let $\mu \in \mathcal{P}^0([0, 1] \times \mathbb{P}(E))$. The generalized Karcher equation is the operator equation

$$
\int_{[0,1] \times \mathbb{P}} X^{1/2} l_s(X^{-1/2}AX^{-1/2})X^{1/2} d\mu(s, A) = 0
$$

for $X \in \mathbb{P}(E)$.

Proposition (Pálfia 2016, Adv. Math.)

*The one parameter family of $\mu$-weighted operator means $L_t(\mu)$ is continuous for $t \in (0, 1]$ with respect to the topology generated by $d_\infty$ (the norm topology).*
The general theory of Lambda operator means

Important observation:

- $t \mapsto L_t(\mu)$ is a decreasing net, i.e. $L_{t_1}(\mu) \leq L_{t_2}(\mu)$ if $0 < t_1 \leq t_2 \leq 1$.

**Theorem (Pálfia 2016, Adv. Math.)**

Let $\mu \in \mathcal{P}^0([0, 1] \times \mathbb{P}(E))$. Then there exists $X_0 \in \mathbb{P}$ such that

$$\lim_{t \to 0+} L_t(\mu) = X_0.$$

**Definition (Lambda operator means)**

Let $\mu \in \mathcal{P}^0([0, 1] \times \mathbb{P}(E))$. Then we denote $\Lambda(\mu) := \lim_{t \to 0+} L_t(\mu)$ and call it the $\mu$-weighted lambda operator mean.
The general theory of Lambda operator means

Theorem (Pálfia 2016, Adv. Math.)

Let $\mu, \mu_1, \mu_2 \in {\mathcal{P}}^0([0, 1] \times \mathbb{P}(E))$. Then

1. $\Lambda(\mu) = A$ if $\mu$ is only supported on $[0, 1] \times \{A\}$;
2. $\Lambda(\mu_1) \leq \Lambda(\mu_2)$ if $\mu_1 \leq \mu_2$ in some generalized sense;
3. $\Lambda(X\mu X^*) = X\Lambda(\mu)X^*$ for any $X \in \text{GL}(E)$;
4. Suppose

$$\int_{[0,1] \times \mathbb{P}} M_{s,t}(X, A) d\mu_1(s, A) \leq \int_{[0,1] \times \mathbb{P}} M_{s,t}(X, A) d\mu_2(s, A).$$

Then $\Lambda(\mu_1) \leq \Lambda(\mu_2)$;

5. If $d\mu_2(s, A) = d\mu_1(s, g(s, A))$ where $g$ is measurable for fixed $s$, then $(1 - u)\Lambda(\mu_1) + u\Lambda(\mu_2) \leq \Lambda((1 - u)\mu_1 + u\mu_2)$ for any $u \in [0, 1]$;
The general theory of Lambda operator means

(6) If $d\mu_2(s, A) = d\mu_1(s, g(s, A))$ where $g$ is measurable for all fixed $s$, then
\[ d_\infty(\Lambda(\mu_1), \Lambda(\mu_2)) \leq \sup_{\mu_2 \text{ puts mass on } \{s\} \times \{A\}} \{ d_\infty(A, g(s, A)) \}; \]

(7) $\Phi(\Lambda(\mu)) \leq \Lambda(\Phi(\mu))$ for any measurable positive unital linear map $\Phi$, where $\Phi(\mu)(s, A) := \mu(s, \Phi^{-1}(A))$.

(8) \[ \left( \int_{[0,1] \times P} A^{-1} d\mu(s, A) \right)^{-1} \leq \Lambda(\mu) \leq \int_{[0,1] \times P} A d\mu(s, A). \]

Theorem (Pálfia 2016, Adv. Math.)

(9) $\Lambda(\mu)$ is the unique solution for $X \in P(E)$ of the generalized Karcher equation
\[ \int_{[0,1] \times P(E)} l_s(X^{-1/2} AX^{-1/2}) d\mu(s, A) = 0. \]
The general theory of Lambda operator means

Since any $f \in \mathcal{L}$ can be written as

$$f(x) = \int_{[0,1]} l_s(x) d\nu(s)$$

we get:

Corollary (Pálfia 2016, Adv. Math.)

Let $f \in \mathcal{L}$ and $\sigma \in \mathcal{P}^0(\mathbb{P}(E))$. Then the equation

$$\int_{\mathbb{P}(E)} f \left( X^{-1/2} A X^{-1/2} \right) d\sigma(A) = 0$$

has a unique solution in $\mathbb{P}$. 
The general theory of Lambda operator means

Theorem (Pálfia 2016, Adv. Math.)

Let \( t \in (0, 1) \) and \( \mu \in \mathcal{P}^0([0, 1] \times \mathbb{P}(E)) \). Then \( L_t(\mu) \) is a lambda operator mean and it is the unique solution of the generalized Karcher equation

\[
\int_{[0,1] \times \mathbb{P}(E)} l_{t+s(1-t)}(X^{-1/2}AX^{-1/2}) d\mu(s, A) = 0.
\]

Corollary (Pálfia 2016, Adv. Math.)

Suppose \( \nu \in \mathcal{P}^0([0, 1]) \) and \( \sigma \in \mathcal{P}^0(\mathbb{P}(E)) \) is supported only on the singletons \( \{A\}, \{B\} \) with \( A, B \in \mathbb{P}(E) \) with weights \( (1 - w), w \in [0, 1] \) respectively. Then \( \Lambda(\nu \times \sigma) \) is a Kubo-Ando mean with respect to the variables \( A, B \in \mathbb{P}(E) \).
The infinite dimensional $L^1$-Karcher mean

The theory of Lambda operator means includes the Karcher mean $\Lambda(\mu)$ for probability measures $\mathcal{P}^0(\mathcal{P}(E))$ but it does not directly generalize to the case of measures with unbounded support. We denote by $\mathcal{P}^1(\mathcal{P}(E))$ the set of all $\tau$-additive probability measures $\mu$ on the Borel sigma algebra generated by $d_\infty$ which are $L^1$-integrable, i.e. $\int_{\mathcal{P}(E)} d_\infty(X, A) d\mu(A) < \infty$ for one (hence all) $X \in \mathcal{P}(E)$. The $L^1$-Wasserstein distance is defined as $W_1(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathcal{P}(E)^2} d_\infty(X, Y) d\gamma(X, Y)$ for $\mu, \nu \in \mathcal{P}^1(\mathcal{P}(E))$ where $\Gamma(\mu, \nu)$ denotes the set of all couplings.

**Proposition**

The topology generated by the Wasserstein metric $W_1(\cdot, \cdot)$ on $\mathcal{P}^1(\mathcal{P}(E))$ agrees with the weak-$\ast$ (also called weak) topology of $\mathcal{P}^1(\mathcal{P}(E))$. Moreover finitely supported probability measures are $W_1$-dense in $\mathcal{P}^1(\mathcal{P}(E))$. 
The infinite dimensional $L^1$-Karcher mean

Using the $W_1$-density of finitely supported probability measures in $\mathcal{P}^1(\mathbb{P}(E))$ we get

**Theorem (Lim-Pálfia 2017)**

*For all $\mu \in \mathcal{P}^1(\mathbb{P}(E))$ there exists a solution of*

\[
\int_{\mathbb{P}(E)} \log X(A) d\mu(A) = 0
\]

*denoted by $\Lambda(\mu)$, which satisfies*

\[
d_{\infty}(\Lambda(\mu), \Lambda(\nu)) \leq W_1(\mu, \nu)
\]

*for all $\nu \in \mathcal{P}^1(\mathbb{P}(E))$.***
The infinite dimensional $L^1$-Karcher mean

Definition (Resolvent operator)

Given $\mu \in \mathcal{P}^1(\mathcal{P}(E))$ we define the resolvent operator for $\lambda > 0$ and $X \in \mathcal{P}(E)$ as

$$J^\mu_\lambda(X) := \Lambda \left( \frac{\lambda}{\lambda + 1} \mu + \frac{1}{\lambda + 1} \delta_X \right),$$

a solution we obtained in the previous theorem of the Karcher equation

$$\frac{\lambda}{\lambda + 1} \int_{\mathcal{P}} \log_Z Ad\mu(A) + \frac{1}{\lambda + 1} \log_Z(X) = 0$$

for $Z \in \mathcal{P}(E)$. 
An ODE flow for the $L^1$-Karcher mean

Theorem (Lim-Pálflia 2017)

For any $X, Y \in \mathcal{P}(E)$ and $t > 0$ the curve

$$S(t)X := \lim_{n \to \infty} \left( J_{t/n}^{\mu} \right)^n (X)$$

exists where the limit is in the $d_\infty$-topology and it is Lipschitz-continuous on compact time intervals $[0, T]$ for any $T > 0$. Moreover it satisfies the contraction property

$$d_\infty (S(t)X, S(t)Y) \leq e^{-t} d_\infty (X, Y),$$

and for $s > 0$ verifies the semigroup property

$$S(t + s)X = S(t)(S(s)X).$$
The infinite dimensional $L^1$-Karcher mean

An ODE flow for the $L^1$-Karcher mean

Theorem (Lim-Pálfia 2017)

Let $\mu \in \mathcal{P}^1(\mathbb{P}(E))$ and $X \in \mathbb{P}(E)$. Then for $t > 0$, the curve $X(t) := S(t)X$ provides a strong solution of the Cauchy problem

$$X(0) := X,$$

$$\dot{X}(t) = \int_\mathbb{P} \log_X(t) \text{Ad}_\mu(A),$$

where the derivative $\dot{X}(t)$ is the Fréchet-derivative.

Theorem (Lim-Pálfia 2017)

Let $\mu \in \mathcal{P}^1(\mathbb{P}(E))$. Then the Karcher equation has a unique solution in $\mathbb{P}(E)$.

The finite dimensional case follows from the theory of gradient flows in CAT($\kappa$) spaces (Ohta-Pálfia 2017, Amer. J. Math.)
Continuous-time i.i.n. for the $L^1$-Karcher mean

Theorem (Lim-Páfia 2017)

Let $\mu \in \mathcal{P}^1(\mathbb{P}(E))$ and let $\{Y_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d.\n$\mathbb{P}(E)$-valued random variables with law $\mu$. Let $\mu_n := \sum_{i=1}^{n} \frac{1}{n} \delta_{Y_i} \in \mathcal{P}^1(\mathbb{P}(E))$ denote the empirical measures. Let $S^\mu(t)$ and $S^\mu_n(t)$ denote the semigroups corresponding to the\nresolvents $J^\mu_\lambda$ and $J^\mu_n_\lambda$ for $t > 0$. Then

$$\lim_{n \to \infty} S^\mu_n(t) = S^\mu(t)$$

uniformly in $d_\infty$ on compact time intervals.

Moreover let $F^\mu_n := J^\delta_{Y_n}_{\rho/n} \circ \cdots \circ J^\delta_{Y_1}_{\rho/n}$ where $J^\delta_{\rho/A}(X) := X \# \frac{\rho}{\rho+1} A$.

Then

$$\lim_{m \to \infty} \left( F^\mu_n \frac{t}{m} \right)^m = S^\mu_n(t)$$

uniformly in $d_\infty$ on compact time intervals for any $n \in \mathbb{N}$. 

On the recent advances in the multivariable theory of operator monotone functions and means

The infinite dimensional $L^1$-Karcher mean
The stochastic partial order for $\mathcal{P}^1(\mathbb{P}(E))$

A set $U \subseteq \mathbb{P}(E)$ is upper if for an $X \in \mathbb{P}(E)$ there exists an $Y \in U$ such that $Y \preceq X$, then $X \in U$. Then the stochastic order for $\mu, \nu \in \mathcal{P}^1(\mathbb{P}(E))$ is defined as $\mu \leq \nu$ if $\mu(U) \leq \nu(U)$ for all upper sets $U \subseteq \mathbb{P}(E)$.

The stochastic order extends the usual positive definite order:

Proposition (Kim 2017, Taiwanese J. Math.)
Let $\mu := \sum_{i=1}^{n} \frac{1}{n} \delta_{A_i}$, $\nu := \sum_{i=1}^{n} \frac{1}{n} \delta_{B_i} \in \mathcal{P}^1(\mathbb{P}(E))$. Then $\mu \leq \nu$ iff $A_i \leq B_i$ for all $1 \leq i \leq n$.

Note: The $L^1$-Karcher mean and Lambda operator means are monotone with respect to the stochastic order.
Free operator monotone functions

Let $\mathcal{L}$ be a fixed Hilbert space. Note: for functions discussed earlier we had $\mathcal{L} = \mathbb{C}$.

Definition (Free function)

A several variable function $F : D(E) \mapsto \mathbb{S}(E \otimes \mathcal{L})$ for a domain $D(E) \subseteq \mathbb{S}(E)^k$ defined for all Hilbert spaces $E$ is called free if for all $E$ and all $A, B \in D(E) \subseteq \mathbb{S}(E)^k$

(1) $F(U^* A_1 U, \ldots, U^* A_k U) = (U^* \otimes I_{\mathcal{L}})F(A_1, \ldots, A_k)(U \otimes I_{\mathcal{L}})$

for all unitary $U^{-1} = U^* \in \mathcal{B}(E)$,

(2) $F\left(\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \ldots, \begin{bmatrix} A_k & 0 \\ 0 & B_k \end{bmatrix}\right) = \\
\begin{bmatrix}
F(A_1, \ldots, A_k) & 0 \\
0 & F(B_1, \ldots, B_k)
\end{bmatrix}.$

It follows: the domain $D(E)$ is closed under direct sums and element-wise unitary conjugation, i.e. $D = (D(E))$ is a free set.
Operator monotone, concave functions

**Definition (Operator monotonicity)**

An free function $F : \mathbb{P}(E)^k \mapsto \mathbb{P}(E \otimes \mathcal{L})$ is operator monotone if for all $X, Y \in \mathbb{P}(E)^k$ s.t. $X \preceq Y$, that is $\forall i \in \{1, \ldots, k\} : X_i \leq Y_i$, we have

$$F(X) \leq F(Y).$$

If this property is verified only (hence up to) $\dim(E) = n$, then $F$ is $n$-monotone. Example: Karcher mean, ALM, BMP, etc.

**Definition (Operator concavity & convexity)**

A free function $F : \mathbb{P}(E)^k \mapsto \mathbb{P}(E \otimes \mathcal{L})$ is operator concave if for all $X, Y \in \mathbb{P}(E)^k$ and $\lambda \in [0, 1]$, we have

$$(1 - \lambda)F(A) + \lambda F(B) \leq F((1 - \lambda)A + \lambda B)$$

Similarly we define $n$-concavity.
Matrix convex sets

Definition (Matrix/Freely convex sets of Wittstock)

A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is a bounded open/closed matrix convex or freely convex set if

(i) $C$ respects direct sums, i.e. if $(X_1, \ldots, X_k) \in C(N)$ and $(Y_1, \ldots, Y_k) \in C(K)$ and $Z_j := X_j \oplus Y_j$, then $(Z_1, \ldots, Z_k) \in C(N \oplus K)$;

(ii) $C$ respects conjugation with isometries, i.e. if $Y \in C(N)$ and $T : K \mapsto N$ is an isometry, then

$$T^*YT = (T^*Y_1T, \ldots, T^*Y_kT) \in C(K).$$

If each $C(E)$ is open/closed then $C = (C(E))$ is open/closed respectively. If each $C(E)$ is bounded then $C = (C(E))$ is bounded.
Matrix convexity of hypographs and operator monotonicity

Given a set $A \subseteq \mathbb{S}(E)$ we define its saturation as

$$\text{sat}(A) := \{X \in \mathbb{S}(E) : \exists Y \in A, Y \geq X\}.$$ 

Similarly for a graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)$, its saturation $\text{sat}(C)$ is the disjoint union of $\text{sat}(C(E))$ for each $E$.

Definition (Hypographs)

Let $F : \mathbb{P}(E)^k \to \mathbb{S}(E)$ be a free function. Then we define its hypograph $\text{hypo}(F)$ as the graded union of the saturation of its image, i.e. $\text{hypo}(F) = (\text{hypo}(F)(E)) := (\{(Y, X) \in \mathbb{S}(E) \times \mathbb{P}(E)^k : Y \leq F(X)\}).$

Theorem (Pálfia 2016)

Let $F : \mathbb{P}(E)^k \to \mathbb{P}(E)$ be a free function. Then its hypograph $\text{hypo}(F)$ is a matrix convex set iff $F$ is operator concave iff $F$ is operator monotone.
Multivariable Loewner’s theorem

The upper operator half-space $\Pi(E)$ consists of $X \in \mathcal{B}(E)$ s.t. $\Im X := \frac{X - X^*}{2i} > 0$.

Theorem (Pálfia 2016)

Let $F : \mathbb{P}(E)^k \mapsto \mathbb{P}(E \otimes \mathcal{L})$ be a free function. Then the following are equivalent

1. $F$ is operator monotone;
2. $F$ is operator concave;
3. There exists a completely positive linear map $\omega : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{L})$ for some auxiliary Hilbert space $\mathcal{H}$, s.t. $F$ is the $\omega$-conditional expectation of the Schur complement of a linear pencil $L_B(X) := B_0 \otimes I + \sum_{i=1}^k B_i \otimes (X_i - I)$ with $B_i \in \hat{\mathbb{P}}(\mathcal{H})$, $B_0 \geq \sum_{i=1}^k B_i$, i.e. $F(X) = (\omega \otimes I)\{S(L_B(X))\}$;
4. $F$ admits a free analytic continuation to the upper operator poly-halfspace $\Pi(E)^k$, mapping $\Pi(E)^k$ to $\Pi(E \otimes \mathcal{L})$ for all $E$. 

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On the recent advances in the multivariable theory of operator monotone functions and means

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Multivariable Loewner’s theorem
Open problems

1. Extend the multivariable Loewner’s theorem to probability measures.
2. Extend the ODE theory of the $L^1$-Karcher mean to Lambda operator means.
3. Determine which Kubo-Ando means are a Lambda operator means.
4. Extend the law of large numbers for the Karcher mean of $L^2$-probability measures in the operator case.
5. Extend the law of large numbers for Lambda operator means of $L^2$-probability measures in the operator or matrix case.
6. Extend any of the above theories to the case of real positive operators. An $A \in \mathcal{B}(E)$ is real positive if $\Re A := \frac{A + A^*}{2} > 0$.

Thank you for your kind attention!