Loewner’s theorem in several variables

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Introduction

In this talk, \( E \) will denote a Hilbert space; \( n, k \) are integers, \( n \) denotes dimension of matrices, \( k \) denotes number of variables.

- \( \mathcal{S}(E) \) denote the space of self-adjoint operators
- \( \mathcal{S}_n \) is its finite n-by-n dimensional part
- \( \mathcal{P} \subseteq \mathcal{S} \) denotes the cone of invertible positive definite and \( \hat{\mathcal{P}} \) the cone of positive semi-definite operators
- \( \mathcal{P}_n \) and \( \hat{\mathcal{P}}_n \) denote the finite dimensional parts

\( \mathcal{S} \) and \( \mathcal{P} \) are partially ordered cones with the positive definite order:

\[ A \leq B \text{ iff } B - A \text{ is positive semidefinite} \]
Loewner’s theorem in several variables

Introduction

Loewner’s theorem

Definition
A real function \( f : (0, \infty) \mapsto \mathbb{R} \) is operator monotone, if \( A \preceq B \) implies \( f(A) \leq f(B) \) for \( A, B \in \mathbb{P}(E) \) and all \( E \).

Theorem (Loewner 1934)
A real function \( f : (0, \infty) \mapsto \mathbb{R} \) is operator monotone if and only if

\[
f(x) = \alpha + \beta x + \int_0^\infty \frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + x} d\mu(\lambda),
\]

where \( \alpha \in \mathbb{R} \), \( \beta \geq 0 \) and \( \mu \) is a unique positive measure on \( [0, \infty) \) such that \( \int_0^\infty \frac{1}{\lambda^2 + 1} d\mu(\lambda) < \infty \); if and only if it has an analytic continuation to the open upper complex half-plane \( \mathbb{H}^+ \), mapping \( \mathbb{H}^+ \) to \( \mathbb{H}^+ \).
Some real operator monotone functions on \( \mathbb{P} \):

- \( x^t \) for \( t \in [0, 1] \);
- \( \log x \);
- \( \frac{x-1}{\log x} \).

**Theorem (a variant of Loewner’s theorem)**

A real function \( f : (0, \infty) \mapsto [0, \infty) \) is operator monotone if and only if

\[
f(x) = \alpha + \beta x + \int_0^\infty \frac{x(1 + \lambda)}{\lambda + x} d\mu(\lambda),
\]

where \( \alpha, \beta \geq 0 \) and \( \mu \) is a unique positive measure on \( (0, \infty) \).

Many different proofs of Loewner’s theorem exists:

- Bendat-Sherman ’55, Hansen ’13, Hansen-Pedersen ’82, Korányi-Nagy ’58, Sparr ’90, Wigner-von Neumann ’54, ...
- According to Barry Simon, the *hard part* of Loewner’s theorem is to obtain the analytic continuation.
Operator connections & means

Definition (Kubo-Ando connection)

A two-variable function $M: \mathbb{P} \times \mathbb{P} \mapsto \mathbb{P}$ is called an operator connection if

1. if $A \leq A'$ and $B \leq B'$, then $M(A, B) \leq M(A', B')$,
2. $CM(A, B)C \leq M(CAC, CBC)$ for all Hermitian $C$,
3. if $A_n \downarrow A$ and $B_n \downarrow B$ then $M(A_n, B_n) \downarrow M(A, B)$,

where $\downarrow$ denotes the convergence in the strong operator topology of a monotone decreasing net.

Theorem (Kubo-Ando 1980)

An $M: \mathbb{P}^2 \mapsto \mathbb{P}$ is an operator connection if and only if

$$M(A, B) = A^{1/2} f \left( A^{-1/2} BA^{-1/2} \right) A^{1/2}$$

where $f: (0, \infty) \mapsto [0, \infty)$ is a real operator monotone function.
Some operator connections on $\mathbb{P}^2$:

- Arithmetic mean: $\frac{A + B}{2}$

- Parallel sum: $A : B = (A^{-1} + B^{-1})^{-1}$

- Geometric mean:
  
  $A\#_t B = A^{1/2} \left( A^{-1/2} BA^{-1/2} \right)^t A^{1/2}$ for $t \in [0, 1]$

The proof of Kubo-Ando’s result relies on the original Loewner theorem.

*Our main question:*

What happens if we have multiple variables in general?
Free functions

Definition (Free function)

A several variable function $F : D(E) \mapsto \mathcal{S}(E)$ for a domain $D(E) \subseteq \mathcal{S}(E)^k$ defined for all Hilbert spaces $E$ is called a free or noncommutative function (NC function) if for all $E$ and all $A, B \in D(E) \subseteq \mathcal{S}(E)^k$

1. $F(U^* A_1 U, \ldots, U^* A_k U) = U^* F(A_1, \ldots, A_k) U$ for all unitary $U^{-1} = U^* \in \mathcal{B}(E)$,

2. $F \left( \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \ldots, \begin{bmatrix} A_k & 0 \\ 0 & B_k \end{bmatrix} \right) = \begin{bmatrix} F(A_1, \ldots, A_k) & 0 \\ 0 & F(B_1, \ldots, B_k) \end{bmatrix}$.

It follows: the domain $D(E)$ is closed under direct sums and element-wise unitary conjugation, i.e. $D = (D(E))$ is a free set.
Operator monotone, concave functions

Definition (Operator monotonicity)

An free function \( F : \mathbb{P}^k \mapsto \mathbb{P} \) is operator monotone if for all \( X, Y \in \mathbb{P}(E)^k \) s.t. \( X \preceq Y \), that is \( \forall i \in \{1, \ldots, k\} : X_i \leq Y_i \), we have

\[
F(X) \preceq F(Y).
\]

If this property is verified only (hence up to) \( \dim(E) = n \), then \( F \) is \( n \)-monotone. Example: Karcher mean, ALM, BMP, etc.

Definition (Operator concavity & convexity)

A free function \( F : \mathbb{P}^k \mapsto \mathbb{P} \) is operator concave if for all \( X, Y \in \mathbb{P}(E)^k \) and \( \lambda \in [0, 1] \), we have

\[
(1 - \lambda)F(X) + \lambda F(Y) \preceq F((1 - \lambda)X + \lambda Y)
\]

Similarly we define \( n \)-concavity.
Karcher mean \( \Lambda(A) \): for \( A \in \mathbb{P}(E)^k \), \( \Lambda(A) \) is the unique positive definite solution of \( \sum_{i=1}^k \log(X^{-1}A_i) = 0 \), if \( \text{dim}(E) < \infty \), then \( \Lambda(A) = \arg \min_{X \in \mathbb{P}(E)} \sum_{i=1}^k d^2(X, A_i) \), where \( d^2(X, Y) = \text{tr}\{\log^2(X^{-1/2} YX^{-1/2})\} \).

Lambda-operator means \( \Lambda_f(A) \): the unique positive definite solution of \( \sum_{i=1}^k f(X^{-1}A_i) = 0 \) for \( A \in \mathbb{P}(E)^k \) and an operator monotone function \( f : (0, \infty) \rightarrow \mathbb{R} \), \( f(1) = 0 \).

Matrix power means \( P_t(A) \): for \( A \in \mathbb{P}(E)^k \) and \( t \in [0, 1] \), \( P_t(A) \) is the unique positive definite solution of \( \sum_{i=1}^k \frac{1}{t} X \#_t A_i = X \).

Inductive mean: \( S(A) := (\cdots (A_1 \#_{1/2} A_2) \#_{1/3} \cdots ) \#_{1/k} A_k \).
Recent multivariable results

For an operator convex free function $F : \mathbb{S}^k \rightarrow \mathbb{S}$ that is rational - hence already free analytic and defined for general tuples of operators by virtue of non-commutative power series expansion - Helton, McCullogh and Vinnikov in 2006 proved a representation formula, that is superficially similar to our formula that we will obtain here later in full generality.

For an operator monotone free function $F : \mathbb{S}^k \rightarrow \mathbb{S}$ Agler, McCarthy and Young in 2012 proved a representation formula valid for commutative tuples of operators, assuming that $F$ as a multivariable real function is continuously differentiable. Using the formula they obtained the analytic continuation of the restricted $F$ to $(H^+)^k$ mapping $(H^+)^k$ to $H^+$. 
Recent multivariable results

In 2013 Pascoe and Tully-Doyle proved that a free function $F : S^k \mapsto S$ that is free analytic, i.e. has a non-commutative power series expansion, thus already defined for general tuples of operators, is operator monotone if and only if it maps the upper operator poly-halfspace $\Pi(E)^k$ to $\Pi(E)$ for all finite dimensional $E$, where $\Pi(E) := \{X \in B(E) : \frac{X - X^*}{2i} > 0\}$.

Our goal is to obtain a result that is valid without any additional assumptions, by establishing the hard part of Loewner’s theorem, thus providing a full generalization.
Proposition
A concave free function $F : \mathbb{P}^k \mapsto \mathbb{S}$ which is locally bounded from below, is continuous in the norm topology.

Proposition (Hansen type theorem)
Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a 2$n$-monotone free function. Then $F$ is $n$-concave, moreover it is norm continuous.

Corollary
An operator monotone free function $F : \mathbb{P}^k \mapsto \mathbb{S}$ is operator concave and norm continuous, moreover it is strong operator continuous on order bounded sets over separable Hilbert spaces $E$. The reverse implication is also true if $F$ is bounded from below:

Theorem
Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be operator concave ($n$-concave) free function. Then $F$ is operator monotone ($n$-monotone).
Supporting linear pencils and hypographs

Definition (Matrix/Freely convex sets of Wittstock)

A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is a bounded open/closed matrix convex or freely convex set if

(i) each $C(E)$ is open/closed;

(ii) $C$ respects direct sums, i.e. if $(X_1, \ldots, X_k) \in C(N)$ and $(Y_1, \ldots, Y_k) \in C(K)$ and $Z_j := X_j \oplus Y_j$, then $(Z_1, \ldots, Z_k) \in C(N \oplus K)$;

(iii) $C$ respects conjugation with isometries, i.e. if $Y \in C(N)$ and $T : K \mapsto N$ is an isometry, then $T^*YT = (T^*Y_1T, \ldots, T^*Y_kT) \in C(K)$;

(iv) each $C(E)$ is bounded.

The above definition has some equivalent characterizations under slight additional assumptions.
Definition
A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is closed with respect to reducing subspaces if for any tuple of operators $(X_1, \ldots, X_k) \in C(E)$ and any corresponding mutually invariant closed subspace $K \subseteq E$, the restricted tuple $(\hat{X}_1, \ldots, \hat{X}_k) \in C(K)$, where each $\hat{X}_i$ is the restriction of $X_i$ to the invariant subspace $K$ for all $1 \leq i \leq k$.

Lemma (Helton, McCullogh 2004)
Suppose that $C = (C(E))$ is a free set, where each $C(E) \subseteq \mathbb{S}(E)^k$, i.e. respects direct sums and unitary conjugation. Then:

(1) If $C$ is closed with respect to reducing subspaces then $C$ is matrix convex if and only if each $C(E)$ is convex in the usual sense of taking scalar convex combinations.

(2) If $C$ is (nonempty and) matrix convex, then $0 = (0, \ldots, 0) \in C(\mathbb{C})$ if and only if $C$ is closed with respect to simultaneous conjugation by contractions.
Given a set $A \subseteq \mathbb{S}(E)$ we define its *saturation* as

$$\text{sat}(A) := \{ X \in \mathbb{S}(E) : \exists Y \in A, Y \geq X \}.$$ 

Similarly for a graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)$, its *saturation* $\text{sat}(C)$ is the disjoint union of $\text{sat}(C(E))$ for each $E$.

**Definition (Hypographs)**

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a free function. Then we define its *hypograph* $\text{hypo}(F)$ as the graded union of the saturation of its image, i.e.

$$\text{hypo}(F) = (\text{hypo}(F)(E)) := \left( \{(Y, X) \in \mathbb{S}(E) \times \mathbb{P}(E)^k : Y \leq F(X) \} \right).$$
Theorem
Let \( F : \mathbb{P}^k \mapsto \mathbb{S} \) be a free function. Then its hypograph \( \text{hypo}(F) \) is a matrix convex set if and only if \( F \) is operator concave.

Corollary
Let \( F : \mathbb{P}^k \mapsto \mathbb{S} \) be a free function. Then its hypograph \( \text{hypo}(F) \) is a matrix convex set if and only if \( F \) is operator monotone.
Linear pencils

Definition (linear pencil)

A linear pencil for $x \in \mathbb{C}^k$ is an expression of the form

$$L_A(x) := A_0 + A_1 x_1 + \cdots + A_k x_k$$

where each $A_i \in \mathbb{S}(K)$ and $\text{dim}(K)$ is the size of the pencil $L_A$. The pencil is monic if $A_0 = I$ and then $L_A$ is a monic linear pencil.

We extend the evaluation of $L_A$ from scalars to operators by tensor multiplication. In particular $L_A$ evaluates at a tuple $X \in \mathbb{S}(N)^k$ as

$$L_A(X) := A_0 \otimes I_N + A_1 \otimes X_1 + \cdots + A_k \otimes X_k.$$

We then regard $L_A(X)$ as a self-adjoint element of $\mathbb{S}(K \otimes N)$ and $L_A$ becomes a free function.
Representation of supporting linear functionals

Suppose $C = (C(E)) \subseteq \mathbb{S}(E)^k$ is a norm closed matrix convex set that is closed with respect to reducing subspaces and $0 \in C(\mathbb{C})$. Then for each boundary point $A \in C(N)$ where $\dim(N) < \infty$, by the Hahn-Banach theorem there exists a continuous supporting linear functional $\Lambda \in (\mathbb{S}(N)^k)^*$ s.t.

$$\Lambda(C(N)) \leq 1 \text{ and } \Lambda(A) = 1$$

and since $\mathbb{S}(N)^* \simeq \mathbb{S}(N)$ we have that for all $X \in \mathbb{S}(N)^k$

$$\Lambda(X) = \sum_{i=1}^{k} \text{tr}\{B_iX_i\}$$

for some $B_i \in \mathbb{S}(N)$. 
Representation of supporting linear functionals

Proposition

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function and let $N$ be a Hilbert space with $\dim(N) < \infty$. Then for each $A \in \mathbb{P}(N)^k$ and each unit vector $v \in N$ there exists a linear pencil

$$L_{F,A,v}(Y, X) := B(F, A, v)_0 \otimes I - vv^* \otimes Y + \sum_{i=1}^{k} B(F, A, v)_i \otimes (X_i - I)$$

of size $\dim(N)$ which satisfies the following properties:

1. $B(F, A, v)_i \in \hat{\mathbb{P}}(N)$ and $\sum_{i=1}^{k} B(F, A, v)_i \leq B(F, A, v)_0$;
2. For all $(Y, X) \in \text{hypo}(F)$ we have $L_{F,A,v}(Y, X) \geq 0$;
3. If $c_1 I \leq A_i \leq c_2 I$ for all $1 \leq i \leq k$ and some fixed real constants $c_2 > c_1 > 0$, then $\text{tr}\{B(F, A, v)_0\} \leq \frac{F(c_2, \ldots, c_2)}{\min(1, c_1)}$. 
Explicit LMI solution formula

**Theorem**

Let $F : \mathbb{P}^k \rightarrow \mathbb{P}$ be an operator monotone function. Then for each $A \in \mathbb{P}(N)^k$ with $\dim(N) < \infty$ and each unit vector $v \in N$

$$F(A)v = v^* B_{0,11}(F, A, v)v \otimes I + \sum_{i=1}^{k} v^* B_{i,11}(F, A, v)v \otimes (A_i - I)v$$

$$- \left\{ (v^* \otimes I) \left[ B_{0,12}(F, A, v) \otimes I + \sum_{i=1}^{k} B_{i,12}(F, A, v) \otimes (A_i - I) \right] \right\}^{-1}$$

$$\times \left[ B_{0,22}(F, A, v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A, v) \otimes (A_i - I) \right]$$

$$\times \left[ B_{0,21}(F, A, v) \otimes I + \sum_{i=1}^{k} B_{i,21}(F, A, v) \otimes (A_i - I) \right]^{-1}$$

$$(v \otimes I) \right\} v$$
and

\[
\left\{ \left[ \overline{B}_{0,22}(A, v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A, v) \otimes A_i \right] \\
- \left[ \overline{B}_{0,21}(A, v) \otimes I + \sum_{i=1}^{k} B_{i,21}(A, v) \otimes A_i \right] \right\} (v^* \otimes v)
\]

\[
= \sum_{j \in I} \left[ \overline{B}_{0,22}(A, v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A, v) \otimes A_i \right] (e_j^* \otimes e_j),
\]

where \( \{e_j\}_{j \in J} \) is an orthonormal basis of \( N \) and

\[
B_{i,11}(F, A, v) := vv^* B_i(F, A, v) vv^*, \\
B_{i,12}(F, A, v) := vv^* B_i(F, A, v) (I - vv^*), \\
B_{i,21}(F, A, v) := (I - vv^*) B_i(F, A, v) vv^*, \\
B_{i,22}(F, A, v) := (I - vv^*) B_i(F, A, v) (I - vv^*)
\]
for all $0 \leq i \leq k$ and $x, y \in \{1, 2\}$.
Moreover if $c_1 I \leq A_i \leq c_2 I$ for all $1 \leq i \leq k$ and some fixed real constants $c_2 > c_1 > 0$, then

$$\operatorname{tr}\{B_0(A, v)\} \leq \frac{F(c_2, \ldots, c_2)}{\min(1, c_1)}.$$  

Definition (Natural map)

A graded map $F : \mathbb{S}(K)^k \times K \mapsto K$ defined for all Hilbert space $K$ is called a *natural map* if it preserves direct sums, i.e.

$$F(X \oplus Y, v \oplus w) = F(X, v) \oplus F(Y, w)$$

for $X \in \mathbb{S}(K_1)^k$, $v \in K_1$ and $Y \in \mathbb{S}(K_2)^k$, $w \in K_2$. 

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Loewner’s theorem in several variables

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Explicit LMI solution formula
For a free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ we define the natural map $\overline{F} : \mathbb{S}(\mathbb{K})^k \times \mathbb{K} \mapsto \mathbb{K}$ for any $\mathbb{K}$ by

$$\overline{F}(X, v) := F(X) v$$

for $X \in \mathbb{S}(\mathbb{K})^k$ and $v \in \mathbb{K}$.

The function below is free, hence induces a natural map:

$$F(X) := v^* B_{0,11} v \otimes I + \sum_{i=1}^{k} v^* B_{i,11} v \otimes X_i$$

$$- (v^* \otimes I) \left[ B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes X_i \right]^{-1}$$

$$\times \left[ B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes X_i \right]^{-1}$$

$$\times \left[ B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes X_i \right] (v \otimes I).$$
Let $S(E) := \{ v \in E : \|v\| = 1 \}$ denote the unit sphere of the Hilbert space $E$. For fixed real constants $c_2 > c_1 > 0$, let

$$\mathbb{P}_{c_1,c_2}(E) := \{ X \in \mathbb{P}(E) : c_1 I \leq X \leq c_2 I \},$$

$$\Omega_{c_1,c_2} := \mathbb{P}_{c_1,c_2}(E)^k \times S(E)$$

and let

$$\mathcal{H} := \bigoplus_{\text{dim}(E) < \infty} \bigoplus_{\omega \in \Omega_{c_1,c_2}} E.$$ 

We equip $\mathcal{H}$ with the inner product

$$x^* y := \sum_{\text{dim}(E) < \infty} \sum_{\omega \in \Omega_{c_1,c_2}} x(\omega)^* y(\omega).$$

Let $\mathcal{B}^+(\mathcal{H})^*$ denote the state space of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}^+(\mathcal{H})_*$ is the normal part.
Definition
Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Now let

$$
\Psi_F(X) := B_{0,11} \otimes I + \sum_{i=1}^{k} B_{i,11} \otimes (X_i - I)
$$

$$
- \left[ B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right]
$$

$$
\times \left[ B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right]^{-1}
$$

$$
\times \left[ B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right]
$$
where

\[
B_{0,xy} := \bigoplus_{\dim(E) < \infty} \bigoplus_{(A,v) \in \Omega_{c_1,c_2}} B_{0,xy}(F, A, v),
\]

\[
B_{i,xy} := \bigoplus_{\dim(E) < \infty} \bigoplus_{(A,v) \in \Omega_{c_1,c_2}} B_{i,xy}(F, A, v)
\]

for \(1 \leq i \leq k\) and \(x, y \in \{1, 2\}\).

**Lemma**

Let \(F : \mathbb{P}^k \mapsto \mathbb{P}\) be an operator monotone function and let \(\dim(E) < \infty\). Let \(A_j \in \mathbb{P}_{c_1,c_2}(E)^k\) and \(v_j \in S(E)\) for \(j \in J\) for some finite index set \(J\). Then there exists a \(w \in S(\mathcal{H})\) such that

\[
F(A_j)v_j = (w^* \otimes I)\Psi_F(A_j)(w \otimes I)v_j
\]

for all \(j \in J\).
Theorem (Multivariable Loewner’s theorem)

Let $F : \mathbb{P}^k \rightarrow \mathbb{P}$ be an operator monotone function. Then there exists a state $\omega \in \mathcal{B}_1^+(\mathcal{H})^*$ such that for all $\dim(E) < \infty$ and $X \in \mathbb{P}(E)^k$ we have

$$F(X) = (\omega \otimes I)(\Psi_F(X)) = \omega(B_{0,11}) \otimes I + \sum_{i=1}^{k} \omega(B_{i,11}) \otimes (X_i - I)$$

$$- (\omega \otimes I) \left\{ \begin{array}{c} B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \\ B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \\ B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \end{array} \right\}^{-1}.$$
The upper operator half-space $\Pi(E)$ consists of $X \in B(E)$ s.t. 
\[ \Im X := \frac{X - X^*}{2i} > 0. \]

**Theorem (Multivariable Loewner’s theorem cont.)**

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be a free function. Then the following are equivalent

1. $F$ is operator monotone;
2. $F$ is operator concave;
3. $F$ is a conditional expectation of the Schur complement of a linear pencil $L_B(X) := B_0 \otimes I + \sum_{i=1}^{k} B_i \otimes (X_i - I)$ over some auxiliary Hilbert space $\mathcal{H}$ with $B_i \in \hat{\mathcal{P}}(\mathcal{H})$, $B_0 \geq \sum_{i=1}^{k} B_i$;
4. $F$ admits a free analytic continuation to the upper operator poly-halfspace $\Pi(E)^k$, mapping $\Pi(E)^k$ to $\Pi(E)$ for all $E$. 
Further results

Let $\mathcal{L}$ be a fixed Hilbert space.

Definition (Free function relaxed)

A several variable function $F : D(E) \mapsto \mathbb{S}(\mathcal{L} \otimes E)$ for a domain $D(E) \subseteq \mathbb{S}(E)^k$ defined for all Hilbert spaces $E$ is called a free if for all $E$ and all $A, B \in D(E) \subseteq \mathbb{S}(E)^k$

(1) \[ F(U^*A_1U, \ldots, U^*A_kU) = (I \otimes U^*)F(A_1, \ldots, A_k)(I \otimes U) \] for all unitary $U^{-1} = U^* \in \mathcal{B}(E),$

(2) \[
F \left( \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \ldots, \begin{bmatrix} A_k & 0 \\ 0 & B_k \end{bmatrix} \right) = \\
\begin{bmatrix} F(A_1, \ldots, A_k) & 0 \\ 0 & F(B_1, \ldots, B_k) \end{bmatrix}.
\]

We may define operator monotonicity of $F$ in the same way: $A \preceq B$ implies $F(A) \preceq F(B)$. 
Theorem (Multivariable Loewner’s theorem II)

Let \( F : \mathcal{P}(E)^k \mapsto \mathcal{P}(\mathcal{L} \otimes E) \) be an operator monotone function. Then there exists a completely positive \( \omega : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{L}) \) such that for all \( \dim(E) < \infty \) and \( X \in \mathcal{P}(E)^k \) we have

\[
F(X) = (\omega \otimes I)(\Psi_F(X)) = \omega(B_{0,11}) \otimes I + \sum_{i=1}^{k} \omega(B_{i,11}) \otimes (X_i - I)
\]

\[
- (\omega \otimes I) \left\{ \left[ B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right]^{-1} \right. \\
\times \left. \left[ B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right] \right. \\
\times \left. \left[ B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right] \right\}
\]


Thank you for your kind attention!